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New Lattice Equation Hierarchies and Darboux Transformation

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Abstract A discrete integrable system and its Hamiltonian structure are generated by use of Tu model. Then, its Darboux transformation is obtained, which can get the expression of the new solutions.

Keywords Discrete integrable system · Discrete zero curvature equation · Hamiltonian structure · Darboux transformation

1 Introduction

In the past twenty years, continuous type integrable systems of the soliton theory has been extensively developed. Meanwhile, the discrete integrable system, treated as models of some physical phenomena, have become the focus of common concern and began to become the most important topic because of its application in Physics, Chemistry and Biology, etc. So it is necessary to pay more attention to the generating of discrete integrable systems and its related properties.

Usually a discrete isospectral problem is given by

$$E\Phi_n = U_n(u_n, \lambda)\Phi_n$$

and the auxiliary problem

$$\Phi_{nt} = V_n(u_n, \lambda)\Phi$$

where the shift operator E is determined by

$$\begin{aligned} &Ef(n) = f(n+1) = f_{n+1}, \\ &E^{-1}f(n) = f(n-1) = f_{n-1}, \\ &(Df)(n) = f(n+1) - f(n) = f_{n+1} - f_n. \end{aligned}$$

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The Gateaux derivative, the variational derivative and the inner product are defined by

$$J'(u)[v] = \frac{\partial}{\partial \varepsilon} J(u+\varepsilon v)|_{\varepsilon=0},$$

$$\frac{\delta \widetilde{H}}{\delta u} = \sum_{m\in z} E^{-1}\left(\frac{\partial H}{\partial u^{(m)}}\right),$$

$$\langle f, g\rangle = \sum_{n\in z}(f(n), g(n)).$$

By use of Tu model [1], a lot of nonlinear integrable lattice equations have been obtained, such as the Toda lattice [1], the Ablowitz-Ladik lattice [2], the Volterra lattice [3], the modified Volterra lattice [4], the differential KdV equation [5], the Blaszak-Marciniak lattice [6, 7] and so on [8–13].

In this paper we construct a discrete isospectral problem, which has three potentials, and get the Lax integrable systems. Using Tu scheme we get its Hamiltonian structure. And then the Darboux transformation is derived to obtain the expression of new solutions.

2 Discrete integrable system

Consider the discrete matrix spectral problem

$$\begin{aligned} &E\Phi_n = U_n\Phi_n, \qquad \lambda_t = 0, \qquad U_n(u,\lambda) = \begin{pmatrix} 0 & r_n \\ s_n & \lambda + p_n \end{pmatrix}, \\ &\Phi_n = \begin{pmatrix} \Phi_n^1 \\ \Phi_n^2 \end{pmatrix}, \qquad u_n = \begin{pmatrix} r_n \\ s_n \\ p_n \end{pmatrix}. \end{aligned} \tag{1}$$

Let

$$\Gamma = \begin{pmatrix} a_n & b_n \\ c_n & -a_n \end{pmatrix}. \tag{2}$$

Solving the stationary discrete zero curvature equation,

$$(E\Gamma)U_n - U_n\Gamma = 0, \tag{3}$$

gives rise to

$$\begin{cases} s_n b_{n+1} - r_n c_n = 0, \\ r_n(a_{n+1}+a_n) + \lambda b_{n+1} + p_n b_{n+1}, \\ -s_n(a_{n+1}+a_n) - \lambda c_n - p_n c_n = 0, \\ r_n c_{n+1} - (\lambda+p_n)(a_{n+1}-a_n) - s_n b_n = 0. \end{cases} \tag{4}$$

Let $a_n = \sum_{m=0}^{\infty} a_n^{(m)}\lambda^{-m}$, $b_n = \sum_{m=0}^{\infty} b_n^{(m)}\lambda^{-m}$, $c_n = \sum_{m=0}^{\infty} c_n^{(m)}\lambda^{-m}$. From (14) we have the following recurrence relation

$$\begin{cases} s_n b_{n+1}^{(m)} - r_n c_n^{(m)} = 0, \\ r_n(a_{n+1}^{(m)} + a_n^{(m)}) + b_{n+1}^{(m+1)} + p_n b_{n+1}^{(m)} = 0, \\ -s_n(a_{n+1}^{(m)} + a_n^{(m)}) - c_n^{(m+1)} - p_n c_n^{(m)} = 0, \\ r_n c_{n+1}^{(m)} - (a_{n+1}^{(m+1)} - a_n^{(m+1)}) - p_n(a_{n+1}^{(m)} - a_n^{(m)}) - s_n b_n^{(m)} = 0. \end{cases} \tag{5}$$

Take $a_n^{(0)}=\frac{1}{2}, b_n^{(0)}=c_n^{(0)}=0$, require $a_j|_{[u]=0}=0, b_j|_{[u]=0}=0, c_j|_{[u]=0}=0, (j\geq 1)$, where $[u]=(u, Eu, E^{-1}u,\ldots)$, the first coefficients are given as following: $a_n^{(1)}=\frac{1}{2}, b_n^{(1)}=-r_{n-1}, c_n^{(1)}=-s_n, a_n^{(2)}=s_n r_{n-1}, b_n^{(2)}=-r_{n-1}+p_{n-1}r_{n-1}, c_n^{(2)}=-s_n+p_n s_n,\ldots$. By this way, the recursion relation (5) determines uniquely a_j, b_j, c_j, $j\geq 1$. Denote

$$\Gamma^{\{m\}}=\sum_{i=0}^{m}\begin{pmatrix} a_n^{(i)}\lambda^{m-i} & b_n^{(i)}\lambda^{m-i} \\ c_n^{(i)}\lambda^{m-i} & -a_n^{(i)}\lambda^{m-i}\end{pmatrix}. \tag{6}$$

Direct calculation reads

$$(E\Gamma^{\{m\}})U_n - U_n\Gamma^{\{m\}}=\begin{pmatrix} 0 & -b_{n+1}^{(m+1)} \\ c_n^{(m+1)} & a_{n+1}^{(m+1)}-a_n^{(m+1)}\end{pmatrix}. \tag{7}$$

Then the discrete zero curvature equation admits the following hierarchy

$$(u_n)_{tm}=\begin{pmatrix} r_n \\ s_n \\ p_n\end{pmatrix}_{tm}=\begin{pmatrix} -b_{n+1}^{(m+1)} \\ c_n^{(m+1)} \\ a_{n+1}^{(m+1)}-a_n^{(m+1)}\end{pmatrix}=J\begin{pmatrix} \frac{a_{n+1}^{(m)}}{r_n} \\ \frac{a_n^{(m)}}{s_n} \\ \frac{c_n^{(m)}}{s_n}\end{pmatrix}, \tag{8}$$

$$J=\begin{pmatrix} 0 & r_n(E+1)s_n & p_n r_n \\ -s_n(1+E^{-1})r_n & 0 & -p_n s_n \\ -p_n r_n & p_n s_n & r_n E s_n - s_n E^{-1} r_n\end{pmatrix}. \tag{9}$$

It is easy to see $J^*=-J.J$ is skew-symmetry operator, i. e. it is satisfy $\langle f, Jg\rangle=-\langle Jf, g\rangle$, in which the inner product is defined ad $\langle f, g\rangle=\sum_{m\in z} f(n)g(n)$.

From (5), we obtain the recurrence operator L as follows

$$L=\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33}\end{pmatrix} \tag{10}$$

where

$$L_{11}=-\frac{1}{r_n}E(E-1)^{-1}p_n r_n, \qquad L_{12}=\frac{1}{r_n}E(E-1)^{-1}p_n s_n,$$
$$L_{13}=\frac{1}{r_n}E(E-1)^{-1}(r_n E s_n - s_n E^{-1} r_n), \qquad L_{21}=-\frac{1}{s_n}(E-1)^{-1}p_n r_n,$$
$$L_{22}=\frac{1}{s_n}p_n s_n, \qquad L_{23}=\frac{1}{s_n}(E-1)^{-1}(r_n E s_n - s_n E^{-1} r_n),$$
$$L_{31}=-r_n, \qquad L_{32}=-s_n, \qquad L_{33}=-p_n.$$

So the system (8) can be written as

$$(u_n)_{tm}=\begin{pmatrix} r_n \\ s_n \\ p_n\end{pmatrix}_{tm}=JL^m\begin{pmatrix} \frac{a_{n+1}^{(1)}}{r_n} \\ \frac{a_n^{(1)}}{s_n} \\ \frac{c_n^{(1)}}{s_n}\end{pmatrix} \tag{11}$$

when $m=1$ in system (11), the system (11) reduces to

$$\begin{cases} r_{nt} = -r_n + p_n r_n, \\ s_{nt} = -s_n + p_n s_n, \\ p_{nt} = s_{n+1} r_n - s_n r_{n-1}. \end{cases} \tag{12}$$

To establish the Hamiltonian structure for system (7), we define

$$V_n = \Gamma U_n^{-1} = \begin{pmatrix} \frac{-(\lambda+p_n)a_n}{r_n s_n} + \frac{b_n}{s_n} & \frac{a_n}{s_n} \\ \frac{-(\lambda+p_n)c_n}{r_n s_n} - \frac{a_n}{r_n} & \frac{c_n}{s_n} \end{pmatrix} \tag{13}$$

and $\langle A, B\rangle = \mathrm{Tr}(AB)$, where A and B are the same order square matrices.

Then we have

$$\begin{aligned} &\frac{\partial U_n}{\partial \lambda} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \frac{\partial U_n}{\partial r_n} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ &\frac{\partial U_n}{\partial s_n} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \frac{\partial U_n}{\partial p_n} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{14}$$

Hence

$$\begin{aligned} &\left\langle V_n, \frac{\partial U_n}{\partial \lambda} \right\rangle = \frac{c_n}{s_n}, \\ &\left\langle V_n, \frac{\partial U_n}{\partial r_n} \right\rangle = -\frac{(\lambda+p_n)c_n}{r_n s_n} - \frac{a_n}{r_n} = \frac{a_{n+1}}{r_n}, \\ &\left\langle V_n, \frac{\partial U_n}{\partial s_n} \right\rangle = \frac{a_n}{s_n}, \\ &\left\langle V_n, \frac{\partial U_n}{\partial p_n} \right\rangle = \frac{c_n}{s_n}. \end{aligned} \tag{15}$$

Substituting (15) into the discrete trace identity

$$\frac{\delta}{\delta u_n} \sum_{k\in z} \left\langle V_n(k), \frac{\partial U_n}{\partial \lambda} \right\rangle = \left(\lambda^{-r} \frac{\partial}{\partial \lambda} \lambda^r\right) \left\langle V_n, \frac{\partial U_n}{\partial u_n^i} \right\rangle, \quad i = 1, 2, 3 \tag{16}$$

yields

$$\begin{pmatrix} \frac{\delta}{\delta r_n} \\ \frac{\delta}{\delta s_n} \\ \frac{\delta}{\delta p_n} \end{pmatrix} \sum_{k\in z} \frac{c_n}{s_n}(k) = \lambda^{-r} \frac{\partial}{\partial \lambda} \lambda^r \begin{pmatrix} \frac{a_{n+1}}{r_n} \\ \frac{a_n}{s_n} \\ \frac{c_n}{s_n} \end{pmatrix}. \tag{17}$$

Comparing the coefficient of λ^{-m-1} yields

$$\begin{pmatrix} \frac{\delta}{\delta r_n} \\ \frac{\delta}{\delta s_n} \\ \frac{\delta}{\delta p_n} \end{pmatrix} \sum_{k\in z} \frac{c_n^{(m+1)}}{s_n}(k) = (r-m) \begin{pmatrix} \frac{a_{n+1}^{(m)}}{r_n} \\ \frac{a_n^{(m)}}{s_n} \\ \frac{c_n^{(m)}}{s_n} \end{pmatrix}. \tag{18}$$

Taking $m=0$ gives $r=0$. Thus

$$\frac{\delta H_m}{\delta u}=\begin{pmatrix}\frac{a_{n+1}^{(m)}}{r_n}\\ \frac{a_n^{(m)}}{s_m}\\ \frac{c_n^{(m)}}{s_n}\end{pmatrix},\qquad H_m=\sum_{k\in}\left(-\frac{c_n^{(m+1)}}{ms_n}\right)(k),\qquad m>0. \tag{19}$$

Hence, the system (11) have the following Hamiltonian structure

$$(u_n)_{tm}=\begin{pmatrix}r_n\\ s_n\\ p_n\end{pmatrix}_{tm}=J\frac{\delta H_m}{\delta u}=JL\frac{\delta H_{m-1}}{\delta u}. \tag{20}$$

3 Darboux transformation and exact solutions

when taking $m=1$ in (6) the spectral problem can be written as

$$Y_{n+1}=U_nY_n,\qquad Y_{nt}=V_nY_n, \tag{21}$$

$$U_n=\begin{pmatrix}0 & r_n\\ s_n & \lambda+p_n\end{pmatrix},\qquad V_n=\Gamma^{\{1\}}=\begin{pmatrix}\frac{1}{2}\lambda+\frac{1}{2} & -r_{n-1}\\ -s_n & -\frac{1}{2}\lambda-\frac{1}{2}\end{pmatrix}. \tag{22}$$

Set $\Phi=(\Phi^1,\Phi^2)^T$, $\Psi=(\Psi^1,\Psi^2)^T$ are the solutions to (21), by use of (Φ,Ψ) we define the transformation matrix

$$T_n=\begin{pmatrix}\lambda+t_{11}(n) & t_{12}(n)\\ \lambda t_{21}(n) & \lambda+t_{22}(n)\end{pmatrix} \tag{23}$$

with

$$\begin{aligned}t_{11}(n)&=\frac{\lambda_1\alpha_2(n)-\lambda_2\alpha_1(n)}{\alpha_1(n)-\alpha_2(n)}, & t_{12}(n)&=\frac{\lambda_1-\lambda_2}{\alpha_2(n)-\alpha_1(n)},\\ t_{21}(n)&=\frac{(\lambda_1-\lambda_2)\alpha_1(n)\alpha_2(n)}{\alpha_1(n)-\alpha_2(n)}, & t_{22}(n)&=\frac{\lambda_1\alpha_1(n)-\lambda_2\alpha_2(n)}{\alpha_2(n)-\alpha_1(n)},\end{aligned} \tag{24}$$

$$\alpha_i(n)=\frac{\Phi_n^2(\lambda_i)-\gamma_i\Psi_n^2(\lambda_i)}{\Phi_n^1(\lambda_i)-\gamma_i\Psi_n^1(\lambda_i)}\quad (i=1,2) \tag{25}$$

here λ_i,γ_i $(i=1,2)$ are the proper parameters make that the terms in (24) and (25) are not zero.

From (23) direct calculation yields

$$\det T_n=(\lambda-\lambda_1)(\lambda-\lambda_2). \tag{26}$$

Assume that there exit a gauge transformation

$$\widetilde{Y}_n=T_nY_n \tag{27}$$

then the spectral problem (23) is transformed to

$$\widetilde{Y}_{n+1}=\widetilde{U}_n\widetilde{Y}_n,\qquad \widetilde{Y}_T=\widetilde{V}_n\widetilde{Y}_n \tag{28}$$

here $\widetilde{U}_n = T_{n+1}U_nT_n^{-1}$, $\widetilde{V}_n = (T_{nt} - T_nV_n)T_n^{-1}$. From (24) and (25) gives

$$\alpha_i(n+1) = \frac{\mu_i(n)}{\nu_i(n)}, \quad i = 1, 2 \tag{29}$$

here

$$\mu_i(n) = s_n + (\lambda_i + p_n)\alpha_i(n), \qquad \nu_i(n) = r_n\alpha_i(n). \tag{30}$$

From (24) and (29) we have

$$\begin{aligned} t_{11}(n+1) &= \frac{\lambda_1\mu_2\nu_1 - \lambda_2\mu_1\nu_2}{\mu_1\nu_2 - \mu_2\nu_1}, \qquad t_{12}(n+1) = \frac{(\lambda_1-\lambda_2)\nu_1\nu_2}{\mu_2\nu_1 - \mu_1\nu_2}, \\ t_{21}(n+1) &= \frac{(\lambda_1-\lambda_2)\mu_1\mu_2}{\mu_1\nu_2 - \mu_2\nu_1}, \qquad t_{22}(n+1) = \frac{\lambda_1\mu_1\nu_2 - \lambda_2\mu_2\nu_1}{\mu_2\nu_1 - \mu_1\nu_2}. \end{aligned} \tag{31}$$

Proposition 1 *The matrix $\widetilde{U}_n = T_{n+1}T_nT_n^{-1}$ has the same form as matrix U_n, that is*

$$\widetilde{U}_n = \begin{pmatrix} 0 & \widetilde{r}_n \\ \widetilde{s}_n & \lambda + \widetilde{p}_n \end{pmatrix} \tag{32}$$

$\widetilde{r}_n, \widetilde{s}_n$ and $\widetilde{p}_n$ are defined as

$$\begin{aligned} \widetilde{r}_n &= r_n + t_{12}(n+1), \\ \widetilde{s}_n &= s_n - t_{21}(n), \\ \widetilde{p}_n &= p_n + t_{22}(n+1) - t_{22}(n). \end{aligned} \tag{33}$$

The transformation $(Y_n; r_n, s_n, p_n \to \widetilde{Y}_n; \widetilde{r}_n, \widetilde{s}_n, \widetilde{p}_n)$ is called a Darboux transformation (DT) of the spectral problem (21), the transformation (33) is called a Backlund transformation (BT) during the potential functions.

Proof Let

$$T_n^{-1} = \frac{T_n^*}{\det T_n}, \qquad T_{n+1}U_nT_n^* = \begin{pmatrix} f_{11}(n,\lambda) & f_{12}(n,\lambda) \\ f_{21}(n,\lambda) & f_{22}(n,\lambda) \end{pmatrix}. \tag{34}$$

It is easy to see that $\lambda^2 f_{11}(n,\lambda), \lambda f_{12}(n,\lambda), \lambda f_{21}(n,\lambda), f_{22}(n,\lambda)$ are 3-order polynomials in λ. Also, we can readily verify that $f_{kl}(\lambda_i, n) = 0$ $(i,k,l = 1,2)$. Based on the above results, we can suppose

$$T_{n+1}U_nT_n^* = (\det T_n)G_n \tag{35}$$

with

$$G_n = \begin{pmatrix} g_{11}^0 & g_{12}^0 \\ g_{21}^0 & \lambda g_{22}^1 + g_{22}^0 \end{pmatrix}. \tag{36}$$

That is

$$T_{n+1}U_n = G_nT_n \tag{37}$$

where g_{ij}^l $(i,j = 1,2; l = 1,0)$ are determined functions in dependent of λ.

By comparing the coefficients of λ^i $(i=1,2)$ in both sides of (37), we obtain

$$\begin{aligned} &g_{11}^0=0, \qquad g_{12}^0=r_n+t_{12}(n+1)=\widetilde{r}_n, \qquad g_{21}^0=s_n-t_{21}(n)=\widetilde{s}_n, \\ &g_{22}^1=1, \qquad g_{22}^0=t_{22}(n+1)+p_n-t_{22}(n)=\widetilde{p}_n. \end{aligned} \tag{38}$$

The proof is completed. □

Proposition 2 *Under the gauge transformation* (27) *and* (33) $\widetilde{V}_n$ *has the following form*

$$\widetilde{V}_n=\begin{pmatrix} \frac{\lambda}{2}+\frac{1}{2} & -\widetilde{r}_{n-1} \\ \widetilde{s}_n & -\frac{\lambda}{2}-\frac{1}{2} \end{pmatrix}. \tag{39}$$

Proof Let

$$T_n^{-1}=\frac{T_n^*}{\det T_n}, \qquad (T_{nt}+T_nV_n)T_n^*=\begin{pmatrix} h_{11}(\lambda,n) & h_{12}(\lambda,n) \\ h_{21}(\lambda,n) & h_{22}(\lambda,n) \end{pmatrix}. \tag{40}$$

It is easy to see that $h_{11}(\lambda,n),\lambda h_{12}(\lambda,n),\lambda h_{21}(\lambda,n),h_{22}(\lambda,n)$ are 3-order polynomials in λ. Also, we can readily verify that $h_{kl}(\lambda_i,n)=0$ $(i,k,l=1,2)$. Based on the above results, we can suppose

$$(T_{nt}+T_nV_n)T_n^*=(\det T_n)F_n \tag{41}$$

with

$$F_n=\begin{pmatrix} \lambda f_{11}^1+f_{11}^0 & f_{12}^0 \\ f_{21}^0 & \lambda f_{22}^1+f_{22}^0 \end{pmatrix}. \tag{42}$$

That is

$$T_{nt}+T_nV_n=F_nT_n \tag{43}$$

where f_{ij}^l, $(i,j=1,2;l=1,0)$ are undetermined functions independent of λ.
By comparing the coefficients of λ^i $(i=0,1,2)$ in both sides of (43), we have

$$\begin{aligned} &f_{11}^1=\frac{1}{2}, \qquad f_{11}^0=\frac{1}{2}, \qquad f_{12}^0=-t_{12}(n)-r_{n-1}=-\widetilde{r}_{n-1}, \\ &f_{21}^0=t_{21}-s_n=-\widetilde{s}_n, \qquad f_{22}^1=-\frac{1}{2}, \qquad f_{22}^0=-\frac{1}{2}. \end{aligned} \tag{44}$$

The proof is completed. □

From above propositions we come to the following conclusion

Proposition 3 *The transformation* $(Y_n;r_n,s_n,p_n\to\widetilde{Y}_n;\widetilde{r}_n,\widetilde{s}_n,\widetilde{p}_n)$ *is the Darboux transformation of the spectral problem* (21), *under the Backlund transformation*

$$\widetilde{r}_n=r_n+t_{12}(n+1), \qquad \widetilde{s}_n=s_n-t_{21}(n), \qquad \widetilde{p}_n=p_n+t_{22}(n+1)-t_{22}(n). \tag{45}$$

The solution r_n,s_n,p_n *are mapped into the new solution* $\widetilde{r}_n,\widetilde{s}_n,\widetilde{p}_n$. *Substituting the trivial solution* $r_n=s_n=p_n=1$ *of* (12) *into* (22) *leads to*

$$Y_{n+1}=\begin{pmatrix} 0 & 1 \\ 1 & \lambda+1 \end{pmatrix}, \qquad Y_{nt}=\begin{pmatrix} \frac{\lambda}{2}+\frac{1}{2} & -1 \\ -1 & -\frac{\lambda}{2}-\frac{1}{2} \end{pmatrix}, \tag{46}$$

its basic solution can be chosen as

$$\Phi_n = \beta^n \exp(t\sqrt{\lambda^2+2\lambda+5})\left(\frac{2}{\lambda+1-\sqrt{\lambda^2+2\lambda+5}}\right),$$
$$\Psi_n = \beta^n \exp(-t\sqrt{\lambda^2+2\lambda+5})\left(\frac{2}{\lambda+1+\sqrt{\lambda^2+2\lambda+5}}\right) \tag{47}$$

with $\beta = (\frac{\lambda+1+\sqrt{\lambda^2+2\lambda+5}}{2})$.

Substituting (47) *into* (25) *we obtain*

$$\alpha_i(n) = \frac{\beta_i^n \exp(2t\sqrt{\xi_i})(\lambda_i+1-\sqrt{\xi_i}) - \gamma_i(-\beta_i^n)(\lambda_i+1+\sqrt{\xi_i})}{2\beta_i^n \exp(2t\sqrt{\xi_i}) - 2\gamma_i(-\beta_i^{-n})}, \tag{48}$$

where $\beta_i = \frac{\lambda_i+1+\sqrt{\lambda_i^2+2\lambda_i+5}}{2}$, $\xi_i = \lambda_i^2+2\lambda_i+5$. *Therefore the new solutions are given as following*

$$\begin{aligned}
\widetilde{r}_n &= 1 + \frac{(\lambda_1-\lambda_2)\alpha_1(n)\alpha_2(n)}{\alpha_1(n)-\alpha_2(n)+(\lambda_2-\lambda_1)\alpha_1(n)\alpha_2(n)},\\
\widetilde{s}_n &= 1 - \frac{(\lambda_1-\lambda_2)\alpha_1(n)\alpha_2(n)}{\alpha_1(n)-\alpha_2(n)},\\
\widetilde{p}_n &= 1 + \frac{\lambda_1\alpha_2(n)-\lambda_2\alpha_1(n)+(\lambda_1^2+\lambda_1-\lambda_2^2-\lambda_2)\alpha_1(n)\alpha_2(n)}{\alpha_1(n)-\alpha_2(n)+(\lambda_2-\lambda_1)\alpha_1(n)\alpha_2(n)}\\
&\quad + \frac{\lambda_1\alpha_1(n)-\lambda_2\alpha_2(n)}{\alpha_2(n)-\alpha_1(n)}.
\end{aligned} \tag{49}$$

Additional, if we choose $\widetilde{r}_n, \widetilde{s}_n, \widetilde{p}_n$ as the seed solution of (12), we can obtain other new explicit solution by use of Darboux transformation again. Repeating the above process again and again we can get multi-soliton solution.

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